HOSSAM GHANEM

(11) 7.6 Hyperbolic functions (A)

Definition	
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\operatorname{csch} x = \dfrac{2}{e^x - e^{-x}}$
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$
$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

$\cosh^2 x - \sinh^2 x = 1$
$1 - \operatorname{sech}^2 x = \tanh^2 x$
$1 + \operatorname{csch}^2 x = \coth^2 x$

$\cosh^2 x = \dfrac{1}{2}(\cosh 2x + 1)$
$\sinh^2 x = \dfrac{1}{2}(\cosh 2x - 1)$

$\sinh 2x = 2 \sinh x \cosh x$
$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

Limits	
$\lim_{x\to\infty} \sinh x = \infty$	$\lim_{x\to-\infty} \sinh x = -\infty$
$\lim_{x\to\infty} \cosh x = \infty$	$\lim_{x\to-\infty} \cosh x = \infty$
$\lim_{x\to\infty} \operatorname{csch} x = 0$	$\lim_{x\to-\infty} \operatorname{csch} x = 0$
$\lim_{x\to\infty} \operatorname{sech} x = 0$	$\lim_{x\to-\infty} \operatorname{sech} x = 0$
$\lim_{x\to\infty} \tanh x = 1$	$\lim_{x\to-\infty} \tanh x = -1$
$\lim_{x\to\infty} \coth x = 1$	$\lim_{x\to-\infty} \coth x = -1$

Example 1	Simplify $\tanh\left(\ln\ \frac{e}{x}\ \right)$	25 August 2005 A

Solution

$$\tanh\left(\ln\ \frac{e}{x}\ \right) = \frac{e^{\ln\frac{e}{x}} - e^{-\ln\frac{e}{x}}}{e^{\ln\frac{e}{x}} + e^{-\ln\frac{e}{x}}} = \frac{\frac{e}{x} - \frac{x}{e}}{\frac{e}{x} + \frac{x}{e}} = \frac{ex\left(\frac{e}{x} - \frac{x}{e}\right)}{ex\left(\frac{e}{x} + \frac{x}{e}\right)} = \frac{e^2 - x^2}{e^2 + x^2}$$

Example 2	Simplify $\cosh\left(\frac{1}{2}\ln(x+1) - \ln 2\right)$ and find its value in a rational form.	18 July 2005 A

Solution

$$\cosh\left(\frac{1}{2}\ln(x+1) - \ln 2\right) = \cosh\ \ln\left(\frac{\sqrt{x+1}}{2}\right)$$

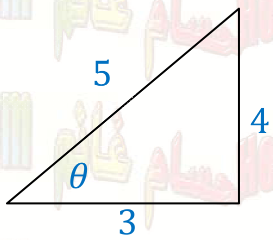
$$= \frac{1}{2}\left[e^{\ln\left(\frac{\sqrt{x+1}}{2}\right)} + e^{-\ln\left(\frac{\sqrt{x+1}}{2}\right)}\right] = \frac{1}{2}\left[\frac{\sqrt{x+1}}{2} + \frac{2}{\sqrt{x+1}}\right] = \frac{1}{2}\cdot\frac{x+1+4}{2\sqrt{x+1}} = \frac{x+5}{4\sqrt{x+1}}$$

Example 3	Find the exact value of $\sec\left(\tan^{-1}\left(\frac{4}{3}\right)\right) + \tanh(\ln 2)$	9 October 1998

Solution

Let $\tan^{-1}\left(\frac{4}{3}\right) = \theta$

$\therefore\ \tan\theta = \frac{4}{3}$



$\therefore \sec\left(\tan^{-1}\left(\frac{4}{3}\right)\right) = \sec(\theta) = \frac{5}{3}$

$$\tanh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{e^{\ln 2} + e^{-\ln 2}} = \frac{2 - \frac{1}{2}}{2 + \frac{1}{2}} == \frac{4-1}{4+1} = \frac{3}{5}$$

$$\therefore \sec\left(\tan^{-1}\left(\frac{4}{3}\right)\right) + \tanh(\ln 2) = \frac{5}{3} + \frac{3}{5} = \frac{25+9}{15} = \frac{34}{15}$$

Example 4	Solve, for x, the equation	$\coth(\ln x) = 3$	22 July 2007

Solution

$\coth(\ln x) = 3 \quad \rightarrow \quad x > 0$

$\frac{e^{\ln x}+e^{-\ln x}}{e^{\ln x}-e^{-\ln x}} = 3$

$\frac{x+\frac{1}{x}}{x-\frac{1}{x}} = 3$

$\frac{x^2+1}{x^2-1} = 3$

$x^2+1 = 3x^2-3$

$2x^2 = 4$

$x^2 = 2$

$x = \sqrt{2}$

Example 5	Prove that	$\frac{\coth(2\ln x)+1}{\coth(2\ln x)-1} = x^4$	25 April 2008

Solution

$$L.H.S = \frac{\coth(2\ln x)+1}{\coth(2\ln x)-1} = \frac{\coth(\ln x^2)+1}{\coth(\ln x^2)-1} = \frac{\sin(\ln x^2)\,(\coth(\ln x^2)+1)}{\sin(\ln x^2)\,(\coth(\ln x^2)-1)} = \frac{\cosh(\ln x^2)+\sin(\ln x^2)}{\mathrm{coch}(\ln x^2)-\sin(\ln x^2)}$$

$$= \frac{\frac{1}{2}\left(e^{\ln x^2}+e^{-\ln x^2}\right)+\frac{1}{2}\left(e^{\ln x^2}-e^{-\ln x^2}\right)}{\frac{1}{2}\left(e^{\ln x^2}+e^{-\ln x^2}\right)-\frac{1}{2}\left(e^{\ln x^2}-e^{-\ln x^2}\right)} = \frac{e^{\ln x^2}}{e^{-\ln x^2}} = \frac{x^2}{\frac{1}{x^2}} = x^4 = R.H.S$$

Example 6	Prove the identity:	$\frac{\operatorname{sech} x}{1-\tanh x} = \cosh x + \sinh x$	12 July 2000 A

Solution

$$L.H.S = \frac{\operatorname{sech} x}{1-\tanh x} = \frac{\cosh x \operatorname{sech} x}{\cosh x\,(1-\tanh x)} = \frac{1}{\cosh x - \sinh x} = \frac{\cosh x + \sinh x}{(\cosh x - \sinh x)(\cosh x + \sinh x)}$$

$$= \frac{\cosh x + \sinh x}{\cosh^2 x - \sinh^2 x} = \cosh x + \sinh x = R.H.S$$

Example 7	Prove that	$\operatorname{sech}(\ln x) + \operatorname{csch}(\ln x) = \frac{4x^3}{x^4-1}$	21 March 2007 A

Solution

$$\operatorname{sech}(\ln x) + \operatorname{csch}(\ln x) = \frac{2}{e^{\ln x}+e^{-\ln x}} + \frac{2}{e^{\ln x}-e^{-\ln x}} = \frac{2}{x+\frac{1}{x}} + \frac{2}{x-\frac{1}{x}} = \frac{2x}{x^2+1} + \frac{2x}{x^2-1}$$

$$= \frac{2x(x^2-1)+2x(x^2+1)}{(x^2+1)(x^2-1)} = \frac{2x(x^2-1+x^2+1)}{(x^2+1)(x^2-1)} = \frac{2x(2x^2)}{(x^2+1)(x^2-1)} = \frac{4x^2}{x^4-1}$$

Example 8 Prove that for all real numbers x 2 March 1993

$$\operatorname{sech}^2 x = \frac{2\operatorname{sech} 2x}{1+\operatorname{sech} 2x}$$

Solution

$$R.H.S = \frac{2\operatorname{sech} 2x}{1+\operatorname{sech} 2x} = \frac{2}{\cosh 2x + 1} = \frac{2}{\frac{1}{2}(e^{2x}+e^{-2x})+1} = \frac{4}{e^{2x}+e^{-2x}+2}$$

$$L.H.S = \operatorname{sech}^2 x = \left(\frac{2}{e^x+e^{-x}}\right)^2 = \frac{4}{e^{2x}+e^{-2x}+2} = R.H.S$$

$$\therefore \operatorname{sech}^2 x = \frac{2\operatorname{sech} 2x}{1+\operatorname{sech} 2x}$$

Example 9 Prove the identity 30 January 2008

$$2\tan^{-1} e^x - \tan^{-1}(\sinh x) = \frac{\pi}{2}, \qquad -\infty < x < \infty$$

Solution

Let $f(x) = 2\tan^{-1} e^x$

$$f^{\backslash}(x) = \frac{2e^x}{e^{2x}+1}$$

Let $g(x) = \tan^{-1}(\sinh x)$

$$g^{\backslash}(x) = \frac{\cosh x}{\sinh^2 x + 1} = \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x} = \frac{2}{e^x+e^{-x}} = \frac{e^x\cdot 2}{e^x(e^x+e^{-x})} = \frac{2e^x}{e^{2x}+1}$$

$f^{\backslash}(x) = g^{\backslash}(x)$

$f(x) = g(x) + c$

$f(x) - g(x) = c$

$2\tan^{-1} e^x - \tan^{-1}(\sinh x) = c$

at $x = 0$

$2\tan^{-1}(1) - \tan^{-1}(\sinh 0) = c$

$2\cdot\frac{\pi}{4} - \tan^{-1}(0) = c$

$\frac{\pi}{2} - 0 = c$

$c = \frac{\pi}{2}$

$\therefore \tan^{-1} e^x - \tan^{-1}(\sinh x) = \frac{\pi}{2}$

Example 10 Given that $x = \ln(\csc\theta + \cot\theta)$ 20 November 2006 A

Show that $\csc\theta = \cosh x$

Solution

$x = \ln(\csc\theta + \cot\theta)$ $\qquad \therefore e^x = \csc\theta + \cot\theta$

$$R.H.S = \cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left(\csc\theta + \cot\theta + \frac{1}{\csc\theta + \cot\theta}\right)$$

$$= \frac{1}{2}\left(\csc\theta + \cot\theta + \frac{1}{\csc\theta + \cot\theta}\cdot\frac{\csc\theta - \cot\theta}{\csc\theta - \cot\theta}\right)$$

$$= \frac{1}{2}\left(\csc\theta + \cot\theta + \frac{\csc\theta - \cot\theta}{\csc^2\theta + \cot^2\theta}\right) = \frac{1}{2}(\csc\theta + \cot\theta + \csc\theta - \cot\theta)$$

$$= \frac{1}{2}(2\csc\theta) = \csc\theta = L.H.S$$

Homework

1	Calculate $\coth(\ln 2)$	23 November 2007 A 27 June 2006 A
2	Verify the following identities: $\cosh(\ln x) - \sinh(\ln x) = \frac{1}{x}$ for , $x > 0$	7 July 1997
3	Show that $\cosh(\ln(\tan x)) = \csc(2x)$, for $0 < x < \frac{\pi}{2}$	28 April 2009 A
4	State whether each of the following statements is true or false-explain your answer $\cosh 4x - \sinh 4x = e^{-4x}$	23 January 2005 A
5	Prove that $\cosh 2x = \cosh^2 x + \sinh^2 x$	29 July 2009 A
6	Prove that $\cosh 2x = \frac{\coth x + \tanh x}{2}$	1 May 1994
7	Solve the equation $e^{3x} + \sinh x = 0$ [2 mark]	31 10 July 2010
8	(2 pts.) Prove that $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$ for all x and y in $\mathbb{R}$.	36 June 6, 2010
9	(3 pts.) Prove that $\tanh(\log_3 x) = \frac{x^{2/\ln 3} - 1}{x^{2/\ln 3} + 1}$	38 Jan. 22, 2011